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Periodic ILW equation with discrete Laplacian

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Abstract
We study an integro-differential equation which generalizes the periodic intermediate long wave (ILW) equation. The kernel of the singular integral involved is an elliptic function written as a second-order difference of the Weierstrass ζ-function. Using Sato's formulation, we show the integrability and construct some special solutions. An elliptic solution is also obtained. We present a conjecture based on a Poisson structure that gives an alternative description of this integrable hierarchy. We note that this Poisson algebra in turn is related to a quantum algebra related to the family of Macdonald difference operators.

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1. Introduction

In this paper, we consider an integrable differential equation with a singular integral term associated with a doubly periodic function. We often classify the known integrable equations with singular integrals according to the periodicity of the kernel functions. Namely, the Benjamin–Ono equation [1, 2] corresponds to the case with no period (rational function) because it has the Hilbert transformation, and the intermediate long wave (ILW) equation [3, 4] corresponds to the singly periodic case (trigonometric function) as hyperbolic cotangent is involved as the kernel. The doubly periodic case was introduced in [5] as a periodic version of the ILW equation.

We aim at constructing an integrable equation which recovers all these equations as special limits, and which also relates to the theory of the Macdonald polynomials [6] (see [7, 8] also). We propose that the kernel of our singular integral is an elliptic function having simple poles at three points $\gamma, 0, -\gamma$ in the fundamental parallelogram with residues $1, -2, 1$ respectively and holomorphic elsewhere.

Let ω_1, ω_2 and γ be complex numbers such that the ratio $\delta = \omega_2/\omega_1$ satisfies $\mathrm{Im}(\delta) > 0$, and $0 < \mathrm{Im}(\gamma) < \mathrm{Im}(\delta)$. Let x and t be real independent variables, and let $\eta(x, t)$ be an analytic function satisfying the periodicity condition $\eta(x + 1, t) = \eta(x, t)$. We consider the integro-differential equation

$$\frac{\partial}{\partial t}\eta(x,t) = \eta(x,t) \cdot \frac{\mathrm{i}\omega_1}{\pi} ⨍_{-1/2}^{1/2} (\Delta_\gamma \zeta)(2\omega_1(y-x)) \cdot \eta(y,t)\,\mathrm{d}y, \tag{1}$$

where $\zeta(x) = \zeta(x; 2\omega_1, 2\omega_2)$ denotes the Weierstrass ζ-function [9] (see (A.1) in the appendix), the discrete Laplacian Δ_γ is defined by $(\Delta_\gamma f)(x) = f(x-\gamma) - 2f(x) + f(x+\gamma)$, and the integral $⨍$ means the Cauchy principal value.

Our purpose in this paper is to study (1) from several viewpoints. First we use the standard method which transforms (1) into a difference-differential form [5, 10–12]. Then we follow the method developed in [13, 14] to utilize the Sato theory to construct an integrable hierarchy which includes (1). Then we study the system of integrals of motion associated with (1) in terms of the Sato theory.

We remark the following. In [10, 12], the conserved densities for the (periodic) ILW were studied by using the Bäcklund transformation. However, we do not know a Bäcklund transformation for (1) at present. It is an open question to find it and compare the approach given in this paper with those classical analyses of conserved quantities.

In the papers [15, 16], a Hamiltonian approach to the ILW equation was pushed forward by using the Gel'fand–Dikij brackets and the bi-Hamiltonian structure. We will develop a Hamiltonian description of (1) in the same spirit as theirs. Our situation, however, might be a little tangled in the following sense. On one hand, it has a direct connection with the Macdonald difference operators [6], or to be more precise, to its elliptic analogue defined through the algebra of Feigin and Odesskii [17] (see [7, 8]). On the other hand, we also attempt to connect the Hamiltonian approach with the Lax formulation of Sato. It is a future problem to understand analogues of the Gel'fand–Dikij brackets and the bi-Hamiltonian structure for (1).

Finally, we make an important comment that the Poisson algebra in this paper has a deep connection with the one found in [18]. The difference analogue of Nth KdV studied by Frenkel has two parameters q and N. It can be found that if we set $q = \mathrm{e}^{2\pi\mathrm{i}\gamma}$ and impose the condition $\delta = N\gamma$, we almost recovers Frenkel's Poisson algebra, but missing the delta function terms which typically appear in the deformed $\mathcal{W}$-algebras (see [19]).

This paper is organized as follows. In section 2, we rewrite (1) in a form of the differential-difference equation. Then we show the ordinary ILW equation with periodicity can be obtained from (1) in the limit $\gamma \to 0$. In section 3, by using the standard Sato theory we present an integrable hierarchy which contains (1) in the lowest order. We study the structure of the integrals of motion in some detail in this setting. Finally, section 4 is devoted to an alternative description based on a Poisson structure derived from a quantum-mechanical integrable model associated with the Macdonald theory, from which we recover the same equation (1), and presumably all the equations given in the hierarchy.

2. Differential-difference form

2.1. Integral operator T

Let T be the integral transformation defined by

$$(\mathsf{T}f)(x) = \frac{\mathrm{i}\omega_1}{\pi} ⨍_{-1/2}^{1/2} (\Delta_\gamma \zeta)(2\omega_1(y-x)) \cdot f(y)\,\mathrm{d}y, \tag{2}$$

then, (1) can be written as $\dot{\eta} = \eta(\mathsf{T}\eta)$.

Decomposing $\eta(x,t)$ by the Plemelj formalism [20], one obtains a differential-difference equation and a bilinear equation from (1). Here and hereafter, we set $p = \mathrm{e}^{2\pi\mathrm{i}\delta}$, $q = \mathrm{e}^{2\pi\mathrm{i}\gamma}$ for simplicity, and let D be a domain in the complex z-plane containing the infinite strip $0 \leqslant \mathrm{Im}(z) \leqslant \mathrm{Im}(\delta)$.

Lemma 2.1. *For any nonzero integer m we have*

$$\mathsf{T}(\mathrm{e}^{2\pi\mathrm{i}mx}) = \frac{(1-q^m)(1-p^m q^{-m})}{1-p^m}\,\mathrm{e}^{2\pi\mathrm{i}mx}, \tag{3}$$

and $\mathsf{T}(1) = 0$.

Proof. Use (A.2) in the appendix. □

Corollary 2.2. *Suppose that an analytic function $f(z)$ on D satisfies the periodicity $f(x+1) = f(x)$, then we have*

$$\mathsf{T}(f(x) - f(x+\delta)) = f(x) - f(x+\gamma) - f(x+\delta-\gamma) + f(x+\delta). \tag{4}$$

Define difference operators T and S acting on the variable x by $Tf(x) = f(x+\delta)$, $Sf(x) = f(x+\gamma)$. For simplicity of display, we also write $\widehat{f} = Tf(x) = f(x+\delta)$, $\overline{f} = Sf(x) = f(x+\gamma)$, $\underline{f} = S^{-1}f(x) = f(x-\gamma)$ and so on. Then we can write (4) as

$$\mathsf{T}(f - \widehat{f}) = (1-S)(1-TS^{-1})f, \tag{5}$$

for example. Setting $g = (1-T)f$, this 'formally' can be expressed as

$$\mathsf{T}g = \frac{(1-S)(1-TS^{-1})}{1-T}g. \tag{6}$$

Proposition 2.3. *Suppose that $w(z)$ is holomorphic on D and satisfies the periodicity $w(x+1) = w(x)$. Set $\eta(x) = w(x) - w(x+\delta) + \eta_0$, where $\eta_0 = \int_{-1/2}^{1/2}\eta(x)\,\mathrm{d}x$ denotes the zero Fourier component. Then we can recast (1) into the difference equation*

$$\frac{\partial}{\partial t}(w - \widehat{w} + \eta_0) = (w - \widehat{w} + \eta_0)(w - \overline{w} - \underline{\widehat{w}} + \widehat{w}). \tag{7}$$

Remark 2.4. Note that we have $\mathrm{d}\eta_0/\mathrm{d}t = 0$ from the assumption on $w(x)$.

Proposition 2.5. *Let ε and η_0 be constants. Assume that τ satisfies the bilinear equation*

$$D_t\widehat{\tau}\cdot\tau = \varepsilon\underline{\widehat{\tau}}\,\overline{\tau} - \eta_0\widehat{\tau}\tau, \tag{8}$$

where D_t denotes the Hirota derivative defined by $D_t f\cdot g = (\partial_t f)g - f(\partial_t g)$. Set $w = -\dfrac{\partial_t\tau}{\tau}$. Then w satisfies the difference equation (7).

Proof. From (8) we have

$$w - \widehat{w} + \eta_0 = -\frac{\partial_t\tau}{\tau} + \frac{\partial_t\widehat{\tau}}{\widehat{\tau}} + \eta_0 = \varepsilon\frac{\underline{\widehat{\tau}}\,\overline{\tau}}{\widehat{\tau}\tau},$$

hence

$$\partial_t\log(w - \widehat{w} + \eta_0) = -\frac{\partial_t\tau}{\tau} + \frac{\partial_t\overline{\tau}}{\overline{\tau}} + \frac{\partial_t\underline{\widehat{\tau}}}{\underline{\widehat{\tau}}} - \frac{\partial_t\widehat{\tau}}{\widehat{\tau}} = (w - \overline{w} - \underline{\widehat{w}} + \widehat{w}).$$

□

2.2. Some special solutions

We give some examples of special solutions to (1), (7) or (8).

Proposition 2.6. *Let n be a positive integer. Let $c_1, \dots, c_n$ be n complex parameters, $k_1, \dots, k_n$ be n integers which are all distinct and nonzero. Set*

$$\begin{aligned} &\tau = \det(f_{l,m})_{1 \leqslant l,m \leqslant n}, \\ &f_{l,m} = \lambda_l^{m-1} + c_l \mu_l^{m-1} \exp(2\pi \mathrm{i}\, k_l x + (\mu_l - \lambda_l)t), \\ &\lambda_l = -\varepsilon \frac{1 - \mathrm{e}^{2\pi \mathrm{i}(\delta-\gamma)k_l}}{1 - \mathrm{e}^{2\pi \mathrm{i}\delta k_l}}, \qquad \mu_l = -\varepsilon \frac{1 - \mathrm{e}^{-2\pi \mathrm{i}(\delta-\gamma)k_l}}{1 - \mathrm{e}^{-2\pi \mathrm{i}\delta k_l}}. \end{aligned} \tag{9}$$

Then this tau satisfies the bilinear equation (8) written for $\eta_0 = \varepsilon$.

The proof of proposition 2.6 will be given in section 3.2.

Remark 2.7. We have

$$\mu_l - \lambda_l = \varepsilon \frac{(1 - \mathrm{e}^{2\pi \mathrm{i}\gamma \mathrm{k_l}})(1 - \mathrm{e}^{2\pi \mathrm{i}(\delta-\gamma)\mathrm{k_l}})}{1 - \mathrm{e}^{2\pi \mathrm{i}\delta \mathrm{k_l}}}. \tag{10}$$

Now we turn to the case of an elliptic solution. Let Δ be a complex number satisfying $\mathrm{Im}(\Delta) > 0$ and define

$$\vartheta_1(u, \Delta) = -\mathrm{i}\, \mathrm{e}^{\pi \mathrm{i}\Delta/4 + \pi \mathrm{i} u} \sum_{m \in \mathbf{Z}} (-1)^m \, \mathrm{e}^{\pi \mathrm{i}\Delta m(m+1) + 2\pi \mathrm{i} m u}. \tag{11}$$

Proposition 2.8. *Let ε and Δ be complex parameters satisfying* $\mathrm{Im}(\Delta) > \mathrm{Im}(\delta)$*, and let k be a nonzero integer. Set*

$$\tau = \vartheta_1(kx + \omega(k)t, \Delta), \tag{12}$$

$$\omega(k) = -\varepsilon \frac{\vartheta_1(k\gamma, \Delta)\vartheta_1(k(\delta-\gamma), \Delta)}{\vartheta_1'(0, \Delta)\vartheta_1(k\delta, \Delta)} \tag{13}$$

and

$$\eta_0 = \omega(k) \left(\frac{\vartheta_1'(k(\gamma-\delta), \Delta)}{\vartheta_1(k(\gamma-\delta), \Delta)} - \frac{\vartheta_1'(k\gamma, \Delta)}{\vartheta_1(k\gamma, \Delta)} \right). \tag{14}$$

Then τ satisfies (8) and

$$\eta = \varepsilon \frac{\widehat{\overline{\tau}}\, \overline{\tau}}{\widehat{\tau}\, \tau} \tag{15}$$

is a special solution to the integro-differential equation (1).

It is straightforward to check this by using addition formulae, so we omit the proof.

Remark 2.9. Note that, Δ (not δ of the period of equation (1)) gives the period of the solution.

2.3. Limit to periodic ILW

Now we consider the limit $\gamma \to 0$, and derive the periodic ILW equatin studied in [5]. Let $\mathcal{T}$ be the integral operator

$$(\mathcal{T} f)(x) = \frac{\mathrm{i}\omega_1}{\pi} \int_{-1/2}^{1/2} \{\zeta(2\omega_1(y-x)) - 2\zeta(\omega_1)(y-x)\} \cdot f(y)\,\mathrm{d}y, \tag{16}$$

then we have

$$\mathcal{T}(\mathrm{e}^{2\pi\mathrm{i}mx}) = -\frac{1}{2}\frac{1+p^m}{1-p^m}\,\mathrm{e}^{2\pi\mathrm{i}mx} \qquad (m \neq 0), \tag{17}$$

and $\mathcal{T}(1) = 0$. Hence we have the formal expression

$$\mathcal{T} g = -\frac{1}{2}\frac{1+T}{1-T} g. \tag{18}$$

Lemma 2.10. *We have the expansion of* T *in* γ *as*

$$\mathsf{T} = -\gamma\partial_x + \gamma^2\mathcal{T}\partial^2 + O(\gamma^3). \tag{19}$$

To have the periodic ILW equation, we need to cancel the leading term $-\gamma\partial_x$. To this end we first perform a Galilean transformation on $\eta(x,t)$ and assume the expansion as $\mathrm{e}^{ct\partial_x}\eta = \varepsilon + \gamma u(x,t) + O(\gamma^2)$, where c and ε are constants. Rescaling t as $t \to \gamma^{-2}t$ and setting $c = \varepsilon\gamma + a\gamma^2 + \cdots$, we have the integro-differential equation of ILW-type for u

$$u_t = au_x - uu_x + \varepsilon\mathcal{T}(u_{xx}). \tag{20}$$

3. Lax formalism

3.1. Sato theory

We introduce an infinite set of independent variables x, r and $\mathbf{t} = (t_1 = t, t_2, t_3, \ldots)$. Define difference operators S, R and T by $Sf(x,r,\mathbf{t}) = f(x+\gamma, r, \mathbf{t})$, $Rf(x,r,\mathbf{t}) = f(x, r+\delta, \mathbf{t})$, $Tf(x,r,\mathbf{t}) = f(x+\delta, r, \mathbf{t})$. For simplicity, we write $Sf = \overline{f}$, $Rf = \widetilde{f}$, $Tf = \widehat{f}$ and so on.

We formulate a version of Sato theory based on the papers [13, 14]. In what follows, we work with a space of operators expressed as formal series in S^{-1}. Let $W = 1 + w_1 S^{-1} + w_2 S^{-2} + \cdots$ be the Sato–Wilson operator. Let $\phi(\lambda)$ and $\kappa(\lambda)$ be Laurent series in λ as

$$\phi(\lambda) = \lambda + \sum_{i\geqslant 0} \phi_{i+1}\lambda^{-i}, \qquad \kappa(\lambda) = \lambda + \sum_{i\geqslant 0} \kappa_{i+1}\lambda^{-i}, \qquad \phi_i, \kappa_i \in \mathbf{C}, \tag{21}$$

and define $\rho(\lambda) = \phi^{-1}(\lambda)$ by the condition

$$\rho(\lambda) = \lambda + \sum_{i\geqslant 0} \rho_{i+1}\lambda^{-i}, \qquad \phi(\rho(\lambda)) = \lambda. \tag{22}$$

We impose the following set of evolution equations on W:

$$\frac{\partial W}{\partial t_j} + W\phi(S)^j = B_j W, \qquad B_j = (W\phi(S)^j W^{-1})_+, \tag{23}$$

$$\widetilde{W}\kappa(S) = CW, \qquad C = (\widetilde{W}\kappa(S)W^{-1})_+, \tag{24}$$

where we have used the standard notation that we write $(A)_+ = \sum_{j=k}^{0} a_j S^{-j}$ for any $A = \sum_{j=k}^{\infty} a_j S^{-j}$.

Remark 3.1. The equality (24) means the truncation of the operator $\widetilde{W}\kappa(S)W^{-1} = (\widetilde{W}\kappa(S)W^{-1})_+$, which involves an infinite system of equations about integrals of motion, and is very much important for our task. Namely, κ_ms play the role of integrals. On the other hand, the parameters ϕ_ms are introduced just to have some linear combination of the time variables t_is. See sections 3.3 and 3.4.

Example of the time evolution (23). We have

$$\partial_{t_1} w_k = w_k(w_1 - \overline{w}_1) - (w_{k+1} - \overline{w}_{k+1}) - \phi_{k+1} - \sum_{j=1}^{k-1} \phi_{j+1} w_{k-j}, \tag{25}$$

etc.

Proposition 3.2. *The compatibility among the evolutions in the* $\mathbf{t}$ *directions and the* r *directions can be described by the Zakharov–Shabat equations*

$$\frac{\partial B_i}{\partial t_j} - \frac{\partial B_j}{\partial t_i} + [B_i, B_j] = 0, \tag{26}$$

$$\frac{\partial C}{\partial t_i} + C B_i - \widetilde{B}_i C = 0. \tag{27}$$

Introduce the wavefunction Ψ defined by

$$\Psi(x, r, \mathbf{t}) = W\rho(\lambda)^{x/\gamma}\kappa(\lambda)^{r/\delta}\, \mathrm{e}^{\xi(\mathbf{t},\lambda)}, \tag{28}$$

$$\xi(\mathbf{t}, \lambda) = \sum_{i=1}^{\infty} t_i \lambda^i. \tag{29}$$

Proposition 3.3. *The wavefunction* Ψ *satisfies*

$$\frac{\partial \Psi}{\partial t_i} = B_i \Psi, \tag{30}$$

$$\widetilde{\Psi} = C\Psi. \tag{31}$$

The operators B_1 and C can be written explicitly as

$$B_1 = S + w_1 - \overline{w}_1 + \phi_1, \qquad C = S + \widetilde{w}_1 - \overline{w}_1 + \kappa_1. \tag{32}$$

Write $w = w_1$ for simplicity, then from Zakharov–Shabat equation (27) we have

$$\frac{\partial}{\partial t}(\widetilde{w} - \overline{w} + \kappa_1) + (w - \widetilde{w} - \overline{w} + \widetilde{\overline{w}})(\widetilde{w} - \overline{w} + \kappa_1) = 0. \tag{33}$$

Definition 3.4. *We call the condition on* W

$$W(x, r + \delta, \mathbf{t}) = W(x + \gamma - \delta, r, \mathbf{t}), \tag{34}$$

($\widetilde{W} = T^{-1}SWTS^{-1}$ *or* $\underline{\widehat{\widetilde{w}}}_k = w_k$*) the reduction condition.*

Proposition 3.5. *Assume that* W *satisfies the reduction condition (34), and define the Lax operator by* $L = TS^{-1}C = WTS^{-1}\kappa(S)W^{-1}$*. Setting* $\eta_0 = \kappa_1, \eta = w - \widehat{w} + \eta_0$*, we have*

$L = T + \eta T S^{-1}$, and w satisfies the difference equation (7). Hence η satisfies (1) under the condition that w is holomorphic on D and periodic $w(x+1) + w(x)$.

Proposition 3.6. *Assuming the reduction condition (34), we have*

$$\frac{\partial \Psi}{\partial t_i} = B_i \Psi, \tag{35}$$

$$L\Psi = \rho(\lambda)^{(\delta-\gamma)/\gamma} \kappa(\lambda)\Psi. \tag{36}$$

3.2. Casorati determinant

As an application of the Sato theory we have studied in the previous subsection, we construct a special solution of (1) in terms of a Casorati determinant. In this subsection, we restrict ourself to the simplest possible situation $\phi(\lambda) = \rho(\lambda) = \lambda$ and $\kappa(\lambda) = \lambda + \varepsilon$ and assume the truncation of the Sato–Wilson operator as

$$W = 1 + w_1 S^{-1} + w_2 S^{-2} + \cdots + w_n S^{-n}. \tag{37}$$

We note that the case $\varepsilon = 0$ necessarily gives us the trivial situation $\eta = 0$. For simplicity we sometimes denote $S^k f(x) = f^{(k)}(x)$, etc.

Consider the linear system $W S^n f = 0$. Let $f_1, f_2, \ldots, f_n$ be the basis of the linear system satisfying the dispersion relation

$$\frac{\partial f_i}{\partial t_j} = S^j f_i, \tag{38}$$

$$R f_i = (S + \varepsilon) f_i. \tag{39}$$

As usual, we explicitly give the basis as

$$f_j = \exp\left(\frac{x}{\gamma}\log\lambda_j + \frac{r}{\delta}\log(\lambda_j + \varepsilon) + \sum_{k\geqslant 1} \lambda_j^k t_k\right) + c_j \exp\left(\frac{x}{\gamma}\log\mu_j + \frac{r}{\delta}\log(\mu_j + \varepsilon) + \sum_{k\geqslant 1}\mu_j^k t_k\right), \tag{40}$$

by introducing the set of parameters $\{\lambda_i, \mu_i, c_i | 1 \leqslant i \leqslant n\}$.

Proposition 3.7. *The operator W uniquely characterized by the linear system $W S^n f_i$ ($i = 1, \ldots, n$) satisfies the evolution equations (23) and (24) written for $\phi(\lambda) = \lambda$, $\rho(\lambda) = \lambda$, $\kappa(\lambda) = \lambda + \varepsilon$.*

For any sequence of integers $\alpha_1, \alpha_2, \ldots, \alpha_n$, we denote the corresponding Casorati determinant by the symbol $|\alpha_1, \alpha_2, \ldots, \alpha_n| = \det\left(f_i^{(\alpha_j)}\right)_{1\leqslant i,j\leqslant n}$. Setting $\tau = |n-1, n-2, \ldots, 0|$, w_is are written as

$$w_k = (-1)^k |n, n-1, \ldots, \check{k}, \ldots, 0|/\tau, \tag{41}$$

where the symbol $\check{k}$ means that the letter k is eliminated. Introduce differential operators p_k by

$$\sum_{k=0}^{\infty} p_k y^k = \exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\partial_{t_m} y^m\right). \tag{42}$$

Proposition 3.8. *We have* $w_k = (p_k\tau)/\tau$, *namely*

$$\sum_{i=0}^{\infty} w_k(x,r,\mathbf{t})\lambda^{-i} = \frac{\tau(x,r,\mathbf{t}-[\lambda^{-1}])}{\tau(x,r,\mathbf{t})}, \tag{43}$$

where we have used the standard notation $[\lambda] = (\lambda/1, \lambda^2/2, \lambda^3/3, \ldots)$.

Note that we especially have $w_1 = -(\partial_{t_1}\tau)/\tau$, $w_n = (-1)^n(S\tau)/\tau$ and $w_{n+1} = w_{n+2} = \cdots = 0$. Hence we have

$$B_1 = S + \frac{\partial_t w_n}{w_n}, \qquad C = S + \varepsilon\frac{\widetilde{w}_n}{w_n}. \tag{44}$$

Proposition 3.9. *If the condition*

$$\frac{\lambda_j+\varepsilon}{\mu_j+\varepsilon} = (\lambda_j/\mu_j)^{(\gamma-\delta)/\gamma}, \tag{45}$$

is satisfied, the reduction condition (34) holds. Such a pair λ_j, μ_j *is parametrized via* k_j *as*

$$\lambda_j = -\varepsilon\frac{1-\mathrm{e}^{2\pi\mathrm{i}(\delta-\gamma)k_j}}{1-\mathrm{e}^{2\pi\mathrm{i}\delta k_j}}, \qquad \mu_j = -\varepsilon\frac{1-\mathrm{e}^{-2\pi\mathrm{i}(\delta-\gamma)k_j}}{1-\mathrm{e}^{-2\pi\mathrm{i}\delta k_j}}. \tag{46}$$

Assuming the reduction condition (34) and setting $L = W(T+\varepsilon TS^{-1})W^{-1} = T+\eta TS^{-1}$, we have the representation of η which satisfies (1) in terms of τ in two ways as

$$\eta = -\frac{\partial_t\tau}{\tau} + \frac{\partial_t\widehat{\tau}}{\widehat{\tau}} + \varepsilon = \varepsilon\frac{\widehat{\underline{\tau}}\,\overline{\tau}}{\widehat{\tau}\tau}. \tag{47}$$

This means that τ satisfies the bilinear equation (8). Thus we have the special solution presented in proposition 2.6.

The wavefunction Ψ associated with this solution reads

$$\Psi = \frac{\tau(x,r,\mathbf{t}-[\lambda^{-1}])}{\tau(x,r,\mathbf{t})}\lambda^{x/\gamma}(\lambda+\varepsilon)^{r/\delta}\,\mathrm{e}^{\xi(\mathbf{t},\lambda)}, \tag{48}$$

which satisfies (under the condition (34))

$$L\Psi = \lambda^{(\delta-\gamma)/\gamma}(\lambda+\varepsilon)\Psi. \tag{49}$$

3.3. Conservation laws

Throughout this subsection, we assume that the reduction condition (34) is satisfied. Introduce the Fourier expansions

$$\eta(x) = \sum_{n\in\mathbf{Z}} \eta_{-n}\,\mathrm{e}^{2\pi\mathrm{i}nx}, \qquad w_k(x) = \sum_{n\in\mathbf{Z}} w_{k,-n}\,\mathrm{e}^{2\pi\mathrm{i}nx}, \tag{50}$$

$$\eta_n = \int_{-1/2}^{1/2}\mathrm{d}x\,\eta(x)\,\mathrm{e}^{2\pi\mathrm{i}nx}, \qquad w_{k,n} = \int_{-1/2}^{1/2}\mathrm{d}x\,w_k(x)\,\mathrm{e}^{2\pi\mathrm{i}nx}. \tag{51}$$

Conjecture 3.10. *Assume the evolution equation in the r-direction (24) and the reduction condition (34).* κ_k*s are given as certain degree k expressions in* $\{\eta_n\}$*. In other words,* $\eta(x)$ *satisfies infinitely many constraint conditions for given* $\{\kappa_k\}$*.*

Example.

$$\kappa_1 = \eta_0, \tag{52}$$

$$\kappa_2 = \sum_{n\neq 0} \frac{p^n q^{-n}}{1-p^n}\eta_n\eta_{-n}, \tag{53}$$

$$\kappa_3 = \sum_{m\neq 0, n\neq 0} \frac{p^m q^{-m}}{1-p^m}\frac{p^n q^{-n}}{1-p^n}\eta_m\eta_{-m+n}\eta_{-n} - \sum_{n\neq 0}\frac{p^n q^{-n}}{(1-p^n)^2}\eta_0\eta_n\eta_{-n}, \tag{54}$$

etc.

We present some explicit calculations to explain what is meant by conjecture 3.10.

Lemma 3.11. *It follows from* $L = T + \eta T S^{-1} = WTS^{-1}\kappa(S)W^{-1}$ *that*

$$\eta \underline{\widehat{w}}_{k-1} = w_k - \widehat{w}_k + \sum_{j=0}^{k-1} \kappa_{j+1} w_{k-j-1}. \tag{55}$$

First we consider the case $k = 1$ in (55), namely $\eta = w_1 - \widehat{w}_1 + \kappa_1$. From this we have the relations among the Fourier modes of w_1, η and κ_1 as

$$w_{1,-n} = \frac{1}{1-p^n}\eta_{-n} \qquad (n\neq 0), \tag{56}$$

$$\kappa_1 = \eta_0, \tag{57}$$

$$w_{1,0} \text{ is free.} \tag{58}$$

Second, by setting $k = 2$ (55) gives us $\eta\underline{\widehat{w}}_1 = w_2 - \widehat{w}_2 + \kappa_1 w_1 + \kappa_2$. Therefore we have

$$w_{2,-n} = \frac{1}{1-p^n}\left(\sum_{l\neq 0}\frac{p^l q^{-l}}{1-p^l}\eta_{-n+l}\eta_{-l} + \eta_{-n}w_{1,0} - \kappa_1\frac{1}{1-p^n}\eta_{-n}\right) \qquad (n\neq 0), \tag{59}$$

$$\kappa_2 = \sum_{l\neq 0}\frac{p^l q^{-l}}{1-p^l}\eta_l\eta_{-l}, \tag{60}$$

$$w_{2,0} \text{ is free,} \tag{61}$$

and so on. In this way, we obtain a series of constraints (52)–(54), etc, besides the relations giving the nonzero Fourier modes of w_ks in terms of η_ks.

Conjecture 3.12. *For* $k = 1, 2, \ldots$ *and nonzero integer* n, $w_{k,-n}$ *is expressed in terms of* η_ks *and* $w_{1,0}, \ldots, w_{k-1,0}$.

Now we present a conjecture that the constraints (52)–(54) etc can be written as sums of multiple integrals whose kernel is simply given by products of theta functions with the period δ:

$$\begin{aligned}\vartheta_1(x;\delta) &= -\mathrm{i}p^{1/8}\,\mathrm{e}^{\pi\mathrm{i}x}\sum_{n\in\mathbf{Z}}(-1)^n p^{n(n+1)/2}\,\mathrm{e}^{2\pi\mathrm{i}nx}\\ &= \mathrm{i}p^{1/8}\,\mathrm{e}^{-\pi\mathrm{i}x}\prod_{n\geqslant 1}(1-p^{n-1}\,\mathrm{e}^{2\pi\mathrm{i}x})(1-p^n\,\mathrm{e}^{-2\pi\mathrm{i}x})(1-p^n).\end{aligned} \tag{62}$$

Note that we have $\vartheta_1' = \vartheta_1'(0) = 2\pi p^{1/8}\prod_{n\geqslant 1}(1-p^n)^3$ and $\vartheta_1(-x) = -\vartheta(x)$.

Definition 3.13. *We define the quantities I_n $(n = 1, 2, \ldots)$ by the multiple integral*

$$I_n = (-1)^{n-1} \frac{1}{n!} \frac{(\vartheta_1')^{n-1} \vartheta_1(n\gamma)}{(2\pi \mathrm{i})^{n-1} \vartheta_1(\gamma)^n} \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} \mathrm{d}x_1 \cdots \mathrm{d}x_n \times \prod_{1 \leqslant i < j \leqslant n} \frac{\vartheta_1(x_i - x_j)^2}{\vartheta_1(x_i - x_j + \gamma)\vartheta_1(x_i - x_j - \gamma)} \cdot \eta(x_1) \cdots \eta(x_n), \tag{63}$$

where we have used the theta function with the period δ and denoted $\vartheta_1(x) = \vartheta_1(x; \delta)$ for short.

Remark 3.14. An explanation is in order here. Similar multiple integrals as in (63) were found in [7] when one of the authors studied the family of Macdonald difference operators in terms of the free field construction. In [8], it is found that the algebra found by Feigin and Odesskii [17] is the underlying algebraic structure for such integrals, including some elliptic extension of the Macdonald theory. From this point of view, one can regard the integrals (63) as a certain classical limit of this. Some detail will be given in section 4.

Conjecture 3.15. *Let the degree of κ_k be k. I_ns are homogeneous polynomials of degree n in κ_ks. Namely the quantities I_ns are conserved under the time evolution described by the integro-differential equation (1).*

Examples.

$$I_1 = \kappa_1, \tag{64}$$

$$I_2 = \kappa_2 + \frac{1}{2} \left(\sum_{n \neq 0} \frac{q^n - p^n q^{-n}}{1 - p^n} \right) \kappa_1^2, \tag{65}$$

$$I_3 = \kappa_3 + \left(\sum_{n \neq 0} \frac{q^n - p^n q^{-2n}}{1 - p^n} \right) \kappa_1 \kappa_2 + \frac{1}{6} \left(\sum_{m,n \neq 0} \frac{q^m - p^m q^{-2m}}{1 - p^m} \frac{q^n - p^n q^{-n}}{1 - p^n} - 2 \sum_{n \neq 0} \frac{q^n (q^n - p^n q^{-2n})}{(1 - p^n)^2} \right) \kappa_1^3, \tag{66}$$

and so on.

3.4. Conserved densities

In this subsection, we continue our study on the structure of the conserved quantities from the point of view of conserved densities.

First let us study the operator $W S^k W^{-1}$ in some detail. Set

$$W S^k W^{-1} = S^k + \sum_{l=1}^{\infty} u_{k,l} S^{k-l}. \tag{67}$$

Lemma 3.16. *Write $\omega_0 = 1$, $u_{k,0} = 1$ for simplicity. Then we have*

$$\sum_{l=0}^{p} u_{k,l} w_{p-l}^{(k-l)} = w_p. \tag{68}$$

Hence $u_{k,l}$ can be expressed as the determinant

$$u_{k,l} = \begin{vmatrix} 1 & 0 & \cdots & 0 & 1 \\ w_1^{(k)} & 1 & & & w_1 \\ w_2^{(k)} & w_1^{(k-1)} & \ddots & & \vdots \\ \vdots & \vdots & & 1 & w_{l-1} \\ w_l^{(k)} & w_{l-1}^{(k-1)} & \cdots & w_1^{(1)} & w_l \end{vmatrix}. \tag{69}$$

As usual, we write res $A = a_0$ for any $A = \sum_{j=k}^{\infty} a_j S^{-j}$.

Proposition 3.17. *For any positive integer k, $u_{k,k} = \mathrm{res}(WS^kW^{-1})$ is a total q-difference.*

Proof. For $j = 1, 2, 3, \ldots,$ set

$$\xi_j = \begin{vmatrix} w_1^{(j)} & 1 & 0 & \cdots & 0 \\ w_2^{(j)} & w_1^{(j-1)} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ w_j^{(j)} & w_{j-1}^{(j-1)} & \cdots & & w_1^{(1)} \end{vmatrix},$$

and set $\xi_0 = 1$. With this notation we can write the determinant (69) written for $l = k$ as

$$u_{k,k} = (-1)^k \sum_{j=1}^{k} (-1)^j \left(w_j \xi_j - w_j^{(j)} \xi_j^{(j)} \right).$$

□

Here are some examples of $u_{k,l}$:

$$u_{1,1} = \mathrm{res}(WSW^{-1}) = (1-S)w_1, \tag{70}$$

$$u_{2,1} = (1-S^2)w_1, \tag{71}$$

$$u_{2,2} = \mathrm{res}(WS^2W^{-1}) = (1-S)(w_2 + \overline{w}_2 - w_1\overline{w}_1), \tag{72}$$

$$u_{3,1} = (1-S^3)w_1, \tag{73}$$

$$u_{3,2} = w_2 - \overline{\overline{\overline{w}}}_2 + \overline{\overline{w}}_1\overline{\overline{\overline{w}}}_1 - w_1\overline{\overline{w}}_1, \tag{74}$$

$$\begin{aligned} u_{3,3} &= \mathrm{res}(WS^3W^{-1}) \\ &= (1-S)(w_3 + \overline{w}_3 + \overline{\overline{w}}_3 - \overline{w}_1 w_2 - \overline{\overline{w}}_1\overline{w}_2 - w_1\overline{\overline{w}}_2 + w_1\overline{w}_1\overline{\overline{w}}_1), \end{aligned} \tag{75}$$

etc.

Now we can state the structure of the residue of $B_k = (W\phi(S)^k W^{-1})_+$.

Proposition 3.18. *For any positive integer n, there exists a difference polynomial E_k of w_ks such that* res B_n = const + $(1-S)E_n$.

Examples.

$$\mathrm{res}\, B_1 = \phi_1 + (1-S)w_1, \tag{76}$$

$$\mathrm{res}\, B_2 = \left(\phi_1^2 + 2\phi_2\right) + 2\phi_1(1-S)w_1 + (1-S)(w_2 + \overline{w}_2 - w_1\overline{w}_1), \tag{77}$$

$$\operatorname{res} B_3 = \left(\phi_1^3 + 6\phi_1\phi_2 + 3\phi_3\right) + 3(\phi_1^2 + \phi_2)(1-S)w_1 + 3\phi_1(1-S)(w_2 + \overline{w}_2 - w_1\overline{w}_1)$$
$$+ (1-S)(w_3 + \overline{w}_3 + \overline{\overline{w}}_3 - \overline{w}_1 w_2 - \overline{\overline{w}}_1\overline{w}_2 - w_1\overline{\overline{w}}_2 + w_1\overline{w}_1\overline{\overline{w}}_1), \tag{78}$$

and so on.

Now we assume the reduction condition (34).

Lemma 3.19. *From the evolution equation* $\partial_{t_m} L = [B_m, L]$*, we have*

$$\partial_{t_m}\eta = \eta\cdot(1 - TS^{-1})\operatorname{res} B_m. \tag{79}$$

Hence from proposition 3.18, we have

$$\partial_{t_m}\eta = \eta\cdot(1 - TS^{-1})(1-S)E_m. \tag{80}$$

Suppose that E_ms are holomorphic on D and periodic $E_m(x+1) = E_m$. Then from corollary 2.2 and (80) we have some quantity H_m which satisfies $\partial_{t_m}\eta = \eta\cdot \mathsf{T}(H_m)$. We have a conjecture which explicitly gives H_m.

Definition 3.20. *We define the densities* $H_n(x)$ $(n = 1, 2, \ldots)$ *by the* $n-1$*-fold multiple integral*

$$H_n(x_1) = (-1)^{n-1}\frac{1}{(n-1)!}\frac{(\vartheta_1')^{n-1}\vartheta_1(n\gamma)}{(2\pi i)^{n-1}\vartheta_1(\gamma)^n}\int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2}\mathrm{d}x_2\cdots\mathrm{d}x_n$$
$$\times\prod_{1\leqslant i<j\leqslant n}\frac{\vartheta_1(x_i - x_j)^2}{\vartheta_1(x_i - x_j+\gamma)\vartheta_1(x_i - x_j - \gamma)}\cdot\eta(x_1)\cdots\eta(x_n), \tag{81}$$

namely we have $nI_n = \int_{-1/2}^{1/2}\mathrm{d}x\, H_n(x)$.

Conjecture 3.21. *By suitably choosing the constants* ϕ_k*s and* c*, we have the equality* $H_m = E_m - \widehat{E}_m + c$*, namely we have*

$$\partial_{t_m}\eta = \eta\cdot\mathsf{T}(H_m). \tag{82}$$

We show some examples. First, we have $E_1 = \omega_1$, and $H_1 = w_1 - \widehat{w}_1 + c$. By choosing the constant as $c = \kappa_1$ we have $H_1 = \eta$. Second, we have $E_2 = w_2 + \overline{w}_2 - w_1\overline{w}_1 + 2\phi_1 w_1$. Thus choosing the constants as

$$c = 2\kappa_2 - \kappa_1(\kappa_1 - 2\phi_1), \tag{83}$$

$$2\phi_1 = \kappa_1\left(1 + \sum_{n\neq 0}\frac{q^n - p^n q^{-n}}{1-p^n}\right), \tag{84}$$

we have

$$E_2 - \widehat{E}_2 + c = -(w_1 - \widehat{w}_1 + \kappa_1)(\overline{w}_1 - \underline{\widehat{w}}_1 + \kappa_1 - 2\phi_1) = H_2. \tag{85}$$

4. Poisson structure

In this section, we study a Poisson structure which is obtained as a certain deformation of the algebra associated with the family of Macdonald difference operators $D_n^r(q,t)$ (see $(3.4)_r$ in VI.3 of [6]), and study the relations with the integro-differential equation (1) and the Lax formulation which we have developed in the previous section.

One of the authors studied [7] the family of Macdonald operators in terms of a Heisenberg algebra and its Fock representation (namely in the case of infinitely many variables $x_1, x_2, \ldots$). In [8], a systematic description of the relation between this and the algebra obtained by Feigin and Odesskii [17] is given (whose trigonometric limit is very precise). In other words, we analyzed the commuting family of operators introduced by Macdonald from the point of view of a pairing between the two kinds of quantum, i.e. noncommutative algebras.

Once we note that the algebra of Feigin and Odesskii is originally constructed over an elliptic curve (which contains three parameters in this most general setting) one may easily find the corresponding elliptic deformation of the commutative family which is acting on the Fock space containing three parameters, say q, t and p.

It is interesting to note that these underlying algebras, the Heisenberg algebra and the Feigin–Odesskii algebra, become commutative in the limit $t \to 1$ with q and p fixed. Hence by setting $t = \mathrm{e}^{\hbar}$ and considering the limit $\hbar \to 0$, one may naturally define Poisson algebras on the corresponding commutative algebras.

Because we lack the space, we skip the derivation and just give the resulting Poisson algebra, then compare the relations with those coming from the Lax formalism. We again use the notations $q = \mathrm{e}^{2\pi\mathrm{i}\gamma}$, $p = \mathrm{e}^{2\pi\mathrm{i}\delta}$ for simplicity of display. The Poisson algebra we study is generated by $\{\lambda_n | n \in \mathbf{Z} \setminus \{0\}\}$ with the Poisson bracket

$$\{\lambda_n, \lambda_m\} = \frac{(1-q^n)(1-p^n q^{-n})}{1-p^n}\delta_{n+m,0}, \tag{86}$$

where $\delta_{m,n}$ denotes the Kronecker delta. Let ε be a constant and set

$$\eta(x) = \sum_{n\in\mathbf{Z}} \eta_n \,\mathrm{e}^{-2\pi\mathrm{i}nx} = \varepsilon \exp\left(\sum_{n\neq 0} \lambda_n \,\mathrm{e}^{-2\pi\mathrm{i}nx}\right). \tag{87}$$

Proposition 4.1. *We have*

$$\{\eta(x), \eta(y)\} = \sum_{n\neq 0} \frac{(1-q^n)(1-p^n q^{-n})}{1-p^n} \,\mathrm{e}^{-2\pi\mathrm{i}n(x-y)} \eta(x)\eta(y). \tag{88}$$

Let us define I_n and $H_n(x)$ by the same equation (63) in definition 3.13 and (81) in definition 3.20 respectively from the quantity defined by (87). Here is a crucial remark: we are not working with the dependent variables described by the Lax formalism, but we are starting from the Poisson algebra and trying to reconstruct the same hierarchy described by the Lax operator L together with the reduction condition (34). First, one can prove the following.

Proposition 4.2. *For any positive integers k and l, we have*

$$\{I_k, I_l\} = 0. \tag{89}$$

Let I_n be our nth Hamiltonian, and set $\partial\eta/\partial t_n = \{I_n, \eta(x)\}$.

Proposition 4.3. *We have* $\partial\eta/\partial t_n = \{I_n, \eta(x)\} = \eta(x)\mathsf{T}(H_n)$.

For example, we have

$$\frac{\partial}{\partial t_1}\eta(x) = \{\eta_0, \eta(x)\} = \eta(x)\sum_{n\neq 0}\frac{(1-q^n)(1-p^nq^{-n})}{1-p^n}\eta_{-n}\,\mathrm{e}^{2\pi\mathrm{i}nx}. \tag{90}$$

The rhs is nothing but $\eta(\mathsf{T}\eta)$, hence we recover the integro-differential equation (1). Note that the time evolution in general takes the same form as we conjecture for the Lax formulation (see conjecture 3.21).

We note that τ also can be presented as a kind of vertex operator. Set

$$\tau(x) = \exp\left(-\sum_{n\neq 0}\frac{p^n}{(1-q^n)(1-p^nq^{-n})}\lambda_n\,\mathrm{e}^{-2\pi\mathrm{i}nx}\right). \tag{91}$$

Then from (87) we have

$$\eta(x) = \varepsilon\frac{\underline{\widehat{\tau}}\,\overline{\tau}}{\widehat{\overline{\tau}}\tau}. \tag{92}$$

We have

$$\{\eta(x), \tau(y)\} = -\sum_{n\neq 0}\frac{1}{1-p^n}\,\mathrm{e}^{-2\pi\mathrm{i}n(x-y)}\eta(x)\tau(y). \tag{93}$$

From this we have

$$\{\eta(x), \tau(y+\delta)\}/\tau(y+\delta) - \{\eta(x), \tau(y)\}/\tau(y) = (\delta(y-x)-1)\eta(x), \tag{94}$$

where we have used the notation $\delta(x) = \sum_n \mathrm{e}^{2\pi\mathrm{i}nx}$. This gives us

$$D_1\widehat{\tau}\cdot\tau = \varepsilon\underline{\widehat{\tau}}\,\overline{\tau} - \eta_0\widehat{\overline{\tau}}\tau, \tag{95}$$

which is exactly the bilinear equation (8).

Appendix. Weierstrass ζ function

The Weierstrass ζ function $\zeta(z) = \zeta(z; 2\omega_1, 2\omega_2)$ is defined by

$$\zeta(z) = \frac{1}{z} + {\sum_{m,n}}'\left\{\frac{1}{z-2m\omega_1-2n\omega_2} + \frac{1}{2m\omega_1+2n\omega_2} + \frac{z}{(2m\omega_1+2n\omega_2)^2}\right\}, \tag{A.1}$$

where the symbol ${\sum}'_{m,n}$ means the summation over $(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}$. We have

$$\zeta(-z) = -\zeta(z),$$
$$\zeta(z+2\omega_1) = \zeta(z) + 2\zeta(\omega_1),$$
$$\zeta(z+2\omega_2) = \zeta(z) + 2\zeta(\omega_2),$$

and the Fourier expansion

$$\zeta(u) = \frac{\zeta(\omega_1)}{\omega_1}u + \frac{\pi}{2\omega_1}\cot\frac{\pi u}{2\omega_1} + \frac{2\pi}{\omega_1}\sum_{n=1}^{\infty}\frac{p^n}{1-p^n}\sin\frac{\pi n u}{\omega_1}, \tag{A.2}$$

where $p = \mathrm{e}^{2\pi\mathrm{i}\omega_2/\omega_1}$.

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